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Structure Synthesis of a Class of 4-DoF and 5-DoF Parallel Manipulators with Identical Limb Structures

Abstract

In this paper, a systematic approach is developed for the structural synthesis of a class of four-degrees-of-freedom (4-DoF) and 5-DoF overconstrained parallel manipulators with identical serial limbs. The theory of screws and reciprocal screws is employed for the analysis of the geometric conditions that must be met by the limbs of such parallel manipulators. Limb structures that can be used for constructing 4-DoF or 5-DoF parallel manipulators are enumerated according to the reciprocity of the twist and wrench systems. The assembly conditions of a parallel manipulator built by using identical C- or F-limbs are discussed. Several 4-DoF and 5-DoF parallel manipulators are sketched as examples.

KEY WORDS—parallel robots, structure synthesis, screw theory, singularity avoidance

1. Introduction

Early research on parallel manipulators concentrated mainly on six-degrees-of-freedom (6-DoF) Steward-Gough-type manipulators. However, a 6-DoF fully parallel manipulator has the limitations of small workspace, difficult direct kinematics, and complex mechanical design. For some industrial applications, a parallel manipulator with fewer than 6-DoF, called a *low-DoF parallel manipulator*, is sufficient. In comparison with a 6-DoF parallel manipulator, a low-DoF parallel manipulator has the advantages of simpler mechanical design, lower manufacturing cost, larger workspace, and simple controller. Therefore, the study of low-DoF parallel manipulators has recently become a main focus among the robotics research community. In particular, 3-DoF parallel manipulators have attracted much attention because of their potential industrial applications (Clavel 1988; Tsai 1996; Tsai, Walsh, and Stamper 1996; Gosselin and Angeles 1989; Karouia and Hervé 2000). However, relatively few papers are available on 4-DoF and 5-DoF parallel manipulators. This is because a general 4-DoF or 5-DoF parallel manipulator cannot be constructed with identical limb structure, as pointed out by Hunt (1983) and Tsai (1998).

Several 4-DoF parallel manipulators with unsymmetrical limbs have been reported recently. Hesselbach et al. (1998) developed a 4-DoF parallel manipulator with two unsymmetrical limbs for cutting convex glass panels. Lenarcic, Stanisic, and Parenti-Castelli (2000) used a 4-DoF parallel mechanism with one PS and three SPS limbs to simulate the shoulder of a humanoid. Rolland (1999) employed parallelograms, similar to the Delta robot, for the construction of two 4-DoF parallel manipulators called Kanuk and Manta, respectively. Both Kanuk and Manta possess one rotational and three translational DoF. Tanev (1998) studied the forward kinematics of a 4-DoF parallel manipulator with one RRPR and two SPS limbs. Wang and Gosselin (1988) investigated the kinematics and singularities of a 4-DoF parallel manipulator with four RUS limbs and one RS passive limb. Chen et al. (2002) presented a 4-DoF parallel manipulator with two PRS and two PSS limbs. A parallel manipulator with unsymmetrical limbs results in unsymmetrical workspace, which may complicate the task planning. Hence, several researchers have made great efforts in designing 4-DoF manipulators with four identical limb structures. Company and Pierrot (1999) presented a new 3T-1R parallel robot. A family of 4-DoF parallel manipulators with identical limbs was investigated by Pierrot and Company (1999) and Pierrot et al. (2001). They used the parallelogram principle to construct two sets of limbs. Each set consists of two identical PUU limbs connected to the moving platform

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by one common revolute joint. Zlatanov and Gosselin (2001) proposed a 4-DoF parallel manipulator structure with four RRRRR limbs. Each limb consists of three intersecting and two parallel revolute joints. Due to the special arrangement of the limbs, the resulting parallel manipulator is an overconstrained mechanism with one translational and three rotational DoF. To the best of our knowledge, this is the first 4-DoF parallel manipulator with four identical serial limbs.

To date, very few 5-DoF parallel manipulators have been proposed. Wang and Gosselin (1997) investigated the kinematics and singularities of a 5-DoF parallel manipulator with five RUS limbs and one US passive limb. Zhang and Gosselin (2001) studied the kinetostatics of a 5-DoF parallel manipulator having a similar limb structure. Lee and Park (1999) investigated the kinematics and dynamics of a 5-DoF parallel manipulator. Merlet, Perng, and Daney (2000) investigated the trajectory planning problem associated with a five-axis machine tool based on a 6-DoF parallel manipulator. An algorithm is proposed to make use of the extra DoF for singularity avoidance. Recently, Huang and Li (2002) presented two 4-DoF and two 5-DoF parallel manipulators with identical limb structures.

In this paper, a systematic approach is presented for the structure synthesis of a class of 4-DoF and 5-DoF parallel manipulators with identical limb structures. First, the screw theory is used to analyze the constraint conditions for such parallel manipulators to possess a given number of DoF. Limb structures that can be used for constructing 4-DoF or 5-DoF parallel manipulators are enumerated. The assembly conditions of a parallel manipulator constructed by using identical C- or F-limbs are discussed. Several 4-DoF and 5-DoF parallel manipulators are sketched as examples.

2. Platform Constraint System Analysis

A parallel manipulator typically consists of a moving platform that is connected to a fixed base by several limbs as illustrated in Figure 1 where only one limb is shown. We assume that each limb in a parallel manipulator is made up of an open-loop chain and the number of limbs equals the number of DoF of the moving platform, such that each limb is driven by one actuator and all actuators can be mounted on or near the fixed base. We also assume that the revolute and prismatic joints are the basic joint types used for each limb. Other types of joint, such as universal and cylindrical joints, can be formed by combining these two basic joints. The kinematic characteristics of parallel manipulators can be best described by using the screw theory (Ball 1900; Hunt 1978; Tsai 1999). In what follows, the theory of screws and reciprocal screws is briefly reviewed, then the constraint system of a parallel manipulator is analyzed, and the constraint conditions that lead to 4-DoF or 5-DoF parallel manipulators are summarized.

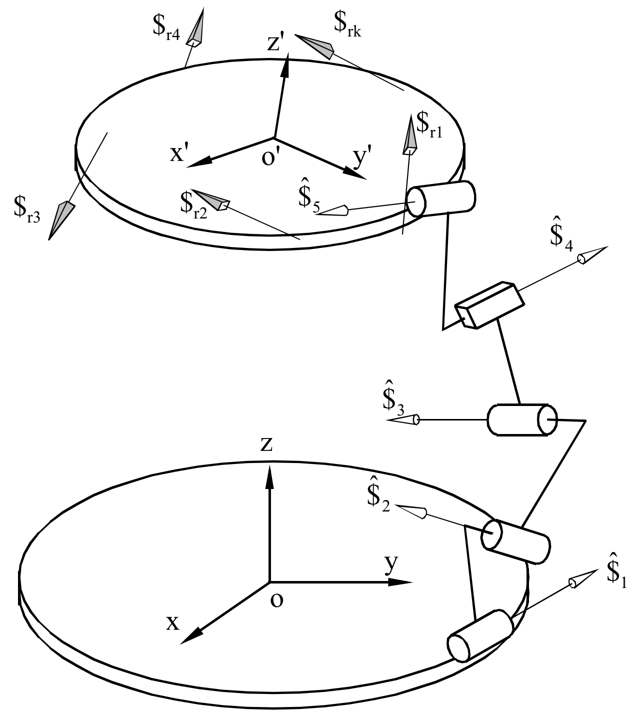

Fig. 1. Platform and limb wrench systems.

2.1. Screw Theory Preliminary

A unit screw $\hat{\$}$ is defined by a straight line with an associated pitch and is conveniently represented by six screw coordinates (Tsai 1999),

$$\hat{\$} = \begin{bmatrix} \mathbf{s} \\ \mathbf{s}_o + \lambda \mathbf{s} \end{bmatrix}, \tag{1}$$

where $\mathbf{s}$ is a unit vector pointing in the direction of the screw axis, $\mathbf{s}_o = \mathbf{r} \times \mathbf{s}$ defines the moment of the screw axis about the origin of a reference frame, $\mathbf{r}$ is the position vector of any point on the screw axis with respect to the reference frame, and λ is the pitch of the screw. If the pitch of a screw is equal to zero, the screw coordinates reduce to

$$\hat{\$} = \begin{bmatrix} \mathbf{s} \\ \mathbf{s}_o \end{bmatrix}. \tag{2}$$

On the other hand, if the pitch of a screw is infinite, the unit screw is defined as

$$\hat{\$} = \begin{bmatrix} \mathbf{0} \\ \mathbf{s} \end{bmatrix}. \tag{3}$$

The unit screw associated with a revolute joint is a screw of zero pitch pointing along the joint axis. The unit screw associated with a prismatic joint is a screw of infinite pitch pointing in the direction of the joint axis.

A screw of intensify ρ is written as $\$ = \rho\hat{\$}$. We call the screw a *twist* if it represents the instantaneous motion of a rigid body, and a *wrench* if it denotes a system of forces and couples acting on a rigid body. In this regard, the first three components of a twist represent the angular velocity and the last three components represent the linear velocity of a point in the rigid body that is instantaneously coincident with the origin of a reference frame. On the other hand, the first three components of a wrench represent the resultant force and the last three components represent the resultant moment about the origin of the reference frame.

Two screws, $\$$ and $\$_r$, are said to be reciprocal if they satisfy the condition

$$\$^{\mathrm{T}}\$_r = 0, \tag{4}$$

where the transpose of a screw is defined as $\$^{\mathrm{T}} = \begin{bmatrix} S_4 & S_5 & S_6 & S_1 & S_2 & S_3 \end{bmatrix}$ such that

$$\begin{aligned} \$^{\mathrm{T}}\$_r = {} & S_4 S_{r1} + S_5 S_{r2} + S_6 S_{r3} \\ & + S_1 S_{r4} + S_2 S_{r5} + S_3 S_{r6}, \end{aligned} \tag{5}$$

where S_i denotes the ith coordinate of the screw $\$$.

The twists associated with all the joints of an n-DoF serial chain form a screw system of order n, called an n-system, provided that all the joint screws are linearly independent. For spatial manipulators, if $n = 6$, there exists no screw that is reciprocal to the six-system of twists. If $n < 6$, there exist 6-n linearly independent wrenches that form a (6-n)-system. Every wrench in the (6-n)-system is reciprocal to the n-system of twists; namely,

$$\hat{\$}_j^{\mathrm{T}}\$_{ri} = 0, \quad (i = 1, 2, \Lambda, 6 - \mathrm{n} \text{ and } j = 1, 2, \Lambda, \mathrm{n}), \tag{6}$$

where $\hat{\$}_j$ is the unit screw associated with the jth joint of a serial chain, and $\$_{ri}$ is ith wrench reciprocal to the n unit joint screws. Given n unit joint screws, $\hat{\$}_j (j = 1, 2, \Lambda, \mathrm{n})$, eq. (6) can be used to solve for a (6-n)-system of reciprocal wrenches. On the other hand, given 6-n linearly independent wrenches, eq. (6) can be used to find an n-system of twists.

2.2. Constraint System of Parallel Manipulators

For convenience, we call the screw system, T_L, associated with the joint screws of a serial limb, the *limb twist system*. As illustrated in Figure 1,

$$T_{\mathrm{L}} = \left[\hat{\$}_1, \hat{\$}_2, \Lambda, \hat{\$}_n\right], \tag{7}$$

where n is the number of 1-DoF joints in a limb. In this study, we limit ourselves to limbs with five basic joints to avoid excessive constraints on the moving platform. We also limit ourselves to those manipulators for which the number of limbs is equal to the number of DoF and all the limbs in a manipulator share the same kinematic structure. Under these assumptions, all the 4-DoF and 5-DoF manipulators considered are overconstrained mechanisms.

The joint screws of a limb form an n-system. The reciprocal screws form a (6-n)-system, W_L, called a *limb wrench system*. We note that each limb contributes one constraint, $\$_{ri}$, to the moving platform as shown in Figure 1 by a pyramid-like arrow. The overall constraint imposed on the moving platform, called a *platform wrench system*, is obtained by a union of the limb wrench systems; that is,

$$W_{\mathrm{P}} = \left[\$_{\mathbf{r1}}, \$_{\mathbf{r2}}, \Lambda, \$_{\mathbf{rk}}\right], \tag{8}$$

where k denotes the number of limbs, $k = 4$ for 4-DoF and 5 for 5-DoF parallel manipulators. The platform twist system, T_P, is reciprocal to the platform wrench. Hence, the platform motion characteristics are completely determined by the nature of the platform wrench system.

If the rank of the platform wrench system is C, the number of DoF of the moving platform, F, is given by

$$F = 6 - C. \tag{9}$$

Hence, the necessary condition for a parallel manipulator to have 4 or 5 DoF is

$$C = \mathrm{Rank}(W_P) = \begin{cases} 2 & \text{for 4-DoF manipulators,} \\ 1 & \text{for 5-DoF manipulators.} \end{cases} \tag{10}$$

To simplify the problem, we assume that all the constraints in the platform wrench system are either pure forces or pure couples. Limbs that provide one constraint force are called F-limbs, and those that provide one constraint couple are called C-limbs. Since we require all limbs in a parallel manipulator to have identical kinematic structure, the resulting platform wrench system consists of either pure forces or pure couples. Following the theory of line geometry (Merlet 1989; Hao and McCarthy 1998), the platform wrench system must satisfy the following conditions:

- C1. If W_P consists of only constraint forces, all the constraint forces must lie in one plane and intersect at a common point for a 4-DoF parallel manipulator, and collinear for a 5-DoF parallel manipulator.
- C2. If W_P consists of only constraint couples, all the couples must be perpendicular to a given direction for a 4-DoF parallel manipulator, and parallel each other for a 5-DoF parallel manipulator.

For parallel manipulators to satisfy the above conditions, it is necessary to identify those limb structures that can provide one pure force or couple of constraint.

3. Enumeration of Limb Structures

In this section, we identify limb structures that can provide either a pure force or a pure couple of constraint.

3.1. Enumeration of F-Limbs

First, limbs that can provide one constraint force on the moving platform are enumerated. Let a unit wrench of zero pitch assume the general form

$$\hat{\$}_r = \begin{bmatrix} \mathbf{s}_r \\ \mathbf{r}_r \times \mathbf{s}_r \end{bmatrix} \quad (11)$$

where $\mathbf{s}_r = \begin{bmatrix} l_r & m_r & n_r \end{bmatrix}^{\mathrm{T}}$ is a unit vector pointing in the direction of the wrench axis, and $\mathbf{r}_r = \begin{bmatrix} x & y & z \end{bmatrix}^{\mathrm{T}}$ is the position vector of a point on the wrench axis. With one such constraint, all feasible twists of a limb form a five-system. Solving eq. (6), we obtain five basis twists:

$$\$_1 = \left[1, \quad 0, \quad 0, \quad -\frac{n_r y - m_r z}{l_r}, \quad 0, \quad 0\right]^{\mathrm{T}}, \quad (12a)$$

$$\$_2 = \left[0, \quad 1, \quad 0, \quad -\frac{l_r z - n_r x}{l_r}, \quad 0, \quad 0\right]^{\mathrm{T}}, \quad (12b)$$

$$\$_3 = \left[0, \quad 0, \quad 1, \quad -\frac{m_r x - l_r y}{l_r}, \quad 0, \quad 0\right]^{\mathrm{T}}, \quad (12c)$$

$$\$_4 = \left[0, \quad 0, \quad 0, \quad -\frac{m_r}{l_r}, \quad 1, \quad 0\right]^{\mathrm{T}}, \quad (12d)$$

$$\$_5 = \left[0, \quad 0, \quad 0, \quad -\frac{n_r}{l_r}, \quad 0, \quad 1\right]^{\mathrm{T}}. \quad (12e)$$

A general twist in the five-system is given by a linear combination of the above five basis twists, namely,

$$\begin{aligned} \$ &= a\$_1 + b\$_2 + c\$_3 + d\$_4 + e\$_5 \\ &= \left[a, \quad b, \quad c, \quad -\frac{dm_r + en_r + f}{l_r}, \quad d, \quad e\right]^{\mathrm{T}}, \end{aligned} \quad (13)$$

where a, b, c, d and e are arbitrary constants that cannot be simultaneously equal to zero, and where $f = a(n_r y - m_r z) + b(l_r z - n_r x) + c(m_r x - l_r y)$. There are two special cases of interest.

Case 1. Letting $\mathbf{s}^{\mathrm{T}}\mathbf{s}_0 = 0$ and $\mathbf{s}^2 = 1$, a zero-pitch unit twist is obtained,

$$\begin{aligned} \$ = \begin{bmatrix} \mathbf{s} \\ \mathbf{s}_0 \end{bmatrix} = \Bigg[a, \quad b, \quad c, \quad & \frac{d(bn_r - cm_r) - cf}{cl_r - an_r} \quad d, \\ & \frac{d(am_r - bl_r) + af}{cl_r - an_r}\Bigg]^{\mathrm{T}}, \end{aligned} \quad (14)$$

where $a^2 + b^2 + c^2 = 1$.

We observe that the direction of the unit twist, $\mathbf{s} = \begin{bmatrix} a & b & c \end{bmatrix}^{\mathrm{T}}$, can be chosen arbitrarily. Once the direction is chosen, the screw axis location is determined by a single parameter, d. Without loss of generality, we may assume that $\mathbf{r}_r = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$. Then $f = 0$, identically, and the moment vector of the screw axis can be written as

$$\mathbf{s}_0 = \mathbf{r} \times \mathbf{s} = -\left(\frac{d}{cl_r - an_r}\right)\mathbf{s}_r \times \mathbf{s}.$$

Hence, if $\mathbf{s} \neq \mathbf{s}_r$, all the zero-pitch twists intersect the given wrench. On the other hand, any twist that is parallel to the given wrench, $\mathbf{s} = \mathbf{s}_r$, can be located anywhere.

Case 2. Letting $a = b = c = 0$, a unit twist of infinite pitch is obtained,

$$\hat{\$} = \begin{bmatrix} \mathbf{0} \\ \mathbf{s} \end{bmatrix} = \left[0, \quad 0, \quad 0, \quad -\frac{dm_r + en_r}{l_r}, \quad d, \quad e\right]^{\mathrm{T}}, \quad (15)$$

where $\left(\frac{dm_r + en_r}{l_r}\right)^2 + d^2 + e^2 = 1$. We observe that $\mathbf{s}^{\mathrm{T}}\mathbf{s}_r = 0$, identically. Hence, all the infinite-pitch twists are perpendicular to the given wrench axis.

In summary, the revolute and prismatic joints of an F-limb must satisfy the following conditions:

C3. All the revolute joints of an F-limb must be intersecting or parallel to the constraint force.

C4. All the prismatic joint axes of an F-limb are perpendicular to the constraint force.

In view of conditions C1 and C3, all the revolute joints of an F-limb must intersect the wrench axis at a common point, or are parallel to the wrench axis. Hence, there are two types of revolute joints. Type-1 revolute joints, denoted as R_1, intersect the wrench axis at a common point. Type-2 revolute joints, denoted as R_2, are parallel to the wrench axis. Furthermore, to uniquely define the location of the constraint force and to avoid redundancy, there must be at least two and at most three type-1 revolute joints.

The maximum number of linearly independent zero-pitch twist in a five-system is five. Hence, the upper limit on the number of revolute joints of an F-limb is five. With consideration of the condition C4, the upper limit on the number of prismatic joints is two. Using revolute (R) and prismatic (P) joints as the basic joint types, we obtain 5R, 4R1P, 3R2P as the feasible four-link, five-jointed, limb configurations. Here a numeral preceding the symbol R or P denotes the number of R or P joints.

Furthermore, for a parallel manipulator to possess finite motion, all type-1 revolute joints must be connected in series. Similarly, all type-2 revolute joints must be connected in series with the possibility of some intermediate prismatic joints. In this regard, the location of the constraint force is determined by the point of intersection of type-1 revolute joints whereas the direction of the constraint force is determined by type-2 revolute joints.

Note that a universal joint (U) is equivalent to two intersecting revolute joints. A cylindrical joint (C) is equivalent to one prismatic joint and a concentric revolute joint. Applying the concept of joint substitution, three-link (four-jointed) and two-link (three-jointed) limb configurations can be derived. All feasible F-limb structures are enumerated in Table 1.

3.2. Enumeration of C-Limbs

Next, we consider those limbs that can provide one constraint couple on the moving platform. Let the unit wrench of infinite pitch assume the form

$$\hat{\$}_r = \begin{bmatrix} 0 \\ \mathbf{s}_r \end{bmatrix} = \begin{bmatrix} 0, & 0, & 0, & l_r, & m_r, & n_r \end{bmatrix}^{\mathrm{T}}, \tag{16}$$

where $l_r^2 + m_r^2 + n_r^2 = 1$. Without losing generality, we assume that $l_r \neq 0$. With one such constraint, all feasible twists of a limb form a five-system. Solving eq. (6), we obtain five basis twists:

$$\$_1 = \begin{bmatrix} -m_r, & l_r, & 0, & 0, & 0, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{17a}$$

$$\$_2 = \begin{bmatrix} -n_r, & 0, & l_r, & 0, & 0, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{17b}$$

$$\$_3 = \begin{bmatrix} 0, & 0, & 0, & 1, & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \tag{17c}$$

$$\$_4 = \begin{bmatrix} 0, & 0, & 0, & 0, & 1, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{17d}$$

$$\$_5 = \begin{bmatrix} 0, & 0, & 0, & 0, & 0 & 1 \end{bmatrix}^{\mathrm{T}}. \tag{17e}$$

A general twist in the five-system is obtained as a linear combination of the above five basis twists; namely,

$$\begin{aligned} \$ &= a\$_1 + b\$_2 + c\$_3 + d\$_4 + e\$_5 = \\ &\begin{bmatrix} -(am_r + bn_r), & al_r, & bl_r, & c & d & e \end{bmatrix}^{\mathrm{T}}, \end{aligned} \tag{18}$$

where a, b, c, d, and e are arbitrary constants, which cannot be simultaneously equal to zero. There are two special cases of interest.

Case 1. Letting $a = b = 0$ and normalizing the direction vector, eq. (18) reduces to a unit twist of infinite pitch,

$$\hat{\$} = \begin{bmatrix} \mathbf{0} \\ \mathbf{s} \end{bmatrix} = \frac{1}{w} \begin{bmatrix} 0, & 0, & 0, & c, & d, & e \end{bmatrix}^{\mathrm{T}}, \tag{19}$$

where $w = \sqrt{c^2 + d^2 + e^2}$. Equation (19) represents a prismatic joint. Since c, d, and e in eq. (19) are arbitrary constants, the prismatic joint axes of a C-limb can be oriented arbitrarily as long as they are linearly independent.

Case 2. Letting $\mathbf{s}^{\mathrm{T}}\mathbf{s}_o = 0$ and $\mathbf{s}^{\mathrm{T}}\mathbf{s} = 1$, eq. (18) reduces to a unit twist of zero pitch,

$$\hat{\$} = \begin{bmatrix} \mathbf{s} \\ \mathbf{s}_0 \end{bmatrix} = \frac{1}{w}$$

$$\begin{bmatrix} -(am_r + bn_r), & al_r, & bl_r, & c, & d, & \dfrac{acm_r - adl_r + bcn_r}{bl_r} \end{bmatrix}^{\mathrm{T}}, \tag{20}$$

where $w = \sqrt{(am_r + bn_r)^2 + a^2 l_r^2 + b^2 l_r^2}$. Equation (20) represents a revolute joint. We observe from eqs. (16) and (20) that

$$\mathbf{s}^{\mathrm{T}}\mathbf{s}_r = 0. \tag{21}$$

We have the following necessary and sufficient conditions for a C-limb:

C5. All the revolute joint axes of a C-limb must be perpendicular to the constraint couple.

C6. The prismatic joint axes, if any, of a C-limb can be oriented arbitrarily as long as they are linearly independent.

In view of conditions C2 and C5, all the revolute joints of a C-limb form two sets of parallel axes. Furthermore, these revolute joints must be arranged in series with the possibility of some intermediate prismatic joints such that the resulting parallel manipulator can undergo finite motion. The number of parallel revolute joints should not exceed three to avoid the existence of redundant joints.

Since the maximum number of linearly independent zero-pitch twists in a five-system is five, the upper limit on the number of revolute joints is five. Furthermore, since the maximum number of linearly independent infinite-pitch twists is equal to three, the upper limit on the number of prismatic joints is three. Using revolute and prismatic joints as the basic joint types for structure synthesis, we obtain 5R, 4R1P, 3R2P, and 2R3P as the feasible four-link, five-jointed, limb configurations.

Applying the concept of joint substitution, three-link (four-jointed) and two-link (three-jointed) limb configurations can be derived. All feasible C-limb structures are enumerated in Table 2, where the subscripts A and B indicate two different directions of rotation.

4. Synthesis of 4-DoF Parallel Manipulators

The main task of structural synthesis of 4-DoF parallel manipulators is to connect a moving platform to a fixed base by four limbs of identical kinematic structure such that after assembly these four limbs provide only two linearly independent constraints to the moving platform. The limb structure can be taken from Tables 1 or 2, or its kinematic inversion. In what follows, we discuss the assembly conditions of 4-DoF parallel manipulators with F- and C-limbs in turn.

4.1. Parallel Manipulators with Four F-Limbs

It follows from condition C1 that the constraint forces provided by the F-limbs must lie in one plane and intersect at a common point. Therefore, the four F-limbs must be arranged in such a way that the following conditions are satisfied:

Table 1. Feasible F-Limbs

Limb Type	Four-link Limbs	Three-link Limbs
5R	$R_1R_1R_1R_2R_2$	$R_1R_1U_{12}R_2$
	$R_1R_1R_2R_2R_2$	$R_1U_{12}R_2R_2$
$4R_1P$	$R_1R_1R_1R_2P$	$R_1R_1U_{12}P$
	$R_1R_1R_1PR_2$	$R_1R_1C_1R_2$
	$R_2R_1R_1R_1P$	$U_{21}R_1R_1P$, $R_2R_1R_1C_1$
	$R_1R_1R_2R_2P$	$R_1U_{12}R_2P$
	$R_1R_1R_2PR_2$	$R_1U_{12}PR_2$
	$R_1R_1PR_2R_2$	$R_1C_1R_2R_2$
	$R_2R_2R_1R_1P$	$R_2U_{21}R_1P$, $R_2R_2R_1C_1$
$3R_2P$	$R_1R_1R_1PP$	$R_1R_1C_1P$
	$R_1R_1R_2PP$	$R_1U_{12}PP$
	$R_1R_1PR_2P$	$R_1C_1R_2P$
	$R_1R_1PPR_2$	$R_1C_1PR_2$
	$R_2R_1R_1PP$	$U_{21}R_1PP$, $R_2R_1C_1P$
	$R_2PR_1R_1P$	$R_2C_1R_1P$, $R_2PR_1C_1$
	$PR_1R_1R_2P$	$C_1R_1R_2P$, $PR_1U_{12}P$

Note that joints arranged in reverse order, known as the kinematic inversion, are not included.

C7. All type-1 revolute joint axes in the four limbs intersect at a common point.

C8. All type-2 revolute joint axes in the four limbs lie in parallel planes.

A parallel manipulator constructed in this way will possess three rotational DoF about the point of intersection of type-1 revolute joints and one translational DoF in a direction perpendicular to a set of parallel planes in which all the constraint forces lie. Figure 2 shows a 4-RRCR parallel manipulator constructed with four F-limbs where the joint axes in each limb are denoted by $\mathbf{s}_{ij}$, where i denotes the limb number and j the joint number. The platform wrench system of the 4-RRCR parallel manipulator consists of four constraint forces intersecting at point O and lying in a plane passing through O and parallel to the fixed platform. Hence, the manipulator possesses three rotational DoF about point O and one translational DoF normal to the base plane. Since type-1 revolute chains are attached to the moving platform, the center of rotation is fixed in the moving platform. The translation of the moving platform simply adjusts the center of rotation with respect to the fixed base. Note that this mechanism is a kinematic inversion of that presented by Zlatanov and Gosselin (2001). Figure 3 illustrates a 4-RRUR parallel manipulator with four F-limbs whose kinematic characteristic is similar to that of the 4-RRCR parallel manipulator.

4.2. Parallel Manipulators with Four C-Limbs

Condition C2 requires the constraint couples provided by the C-limbs to be perpendicular to a given direction. Therefore, the four C-limbs must be arranged such that the following conditions are satisfied:

C9. The revolute joints in each limb form two sets of parallel axes whose common normal defines the direction of a constraint couple.

C10. The four common normals of a C-limb form two sets of parallel lines.

A parallel manipulator constructed in this way will possess 4-DoF, one rotational and three translational DoF. The rotational axis of the moving platform is perpendicular to the constraint couples. Figure 4 illustrates a 4-RUC parallel manipulator constructed with four C-limbs. Two linearly independent constraint couples, $\$_{r1}$ and $\$_{r2}$, satisfy the conditions: $\boldsymbol{s}_{r1}//(\boldsymbol{s}_{12} \times \boldsymbol{s}_{13})$ and $\boldsymbol{s}_{r2}//(\boldsymbol{s}_{22} \times \boldsymbol{s}_{23})$. Hence, the moving platform possesses three translational DoF and one rotational DoF about the axis parallel to the vector $\mathbf{s}_{r1} \times \mathbf{s}_{r2}$. Note that the $\mathbf{s}_{21}$ and $\mathbf{s}_{41}$ axes are parallel to each other but do not lie in the base plane, in order to prevent the platform wrench system degenerating when the moving and base platforms are parallel.

It is noted that, if a 4-DoF parallel manipulator is constructed with four identical F- or C-limbs, it is not possible to obtain two translational and two rotational DoF combinations.

5. Synthesis of 5-DoF Parallel Manipulators

The synthesis of 5-DoF parallel manipulator involves an arrangement of five C- or F-limbs of identical kinematic

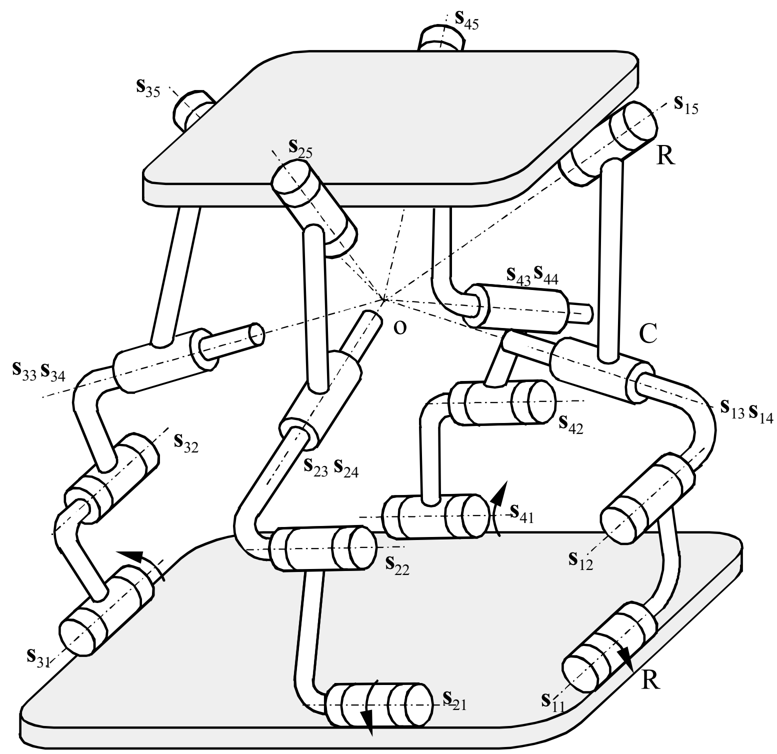

Fig. 2. A 4-DoF manipulator constructed with 4-RRCR F-limbs ($\mathbf{s}_{i1}//\mathbf{s}_{i2}$, $\mathbf{s}_{i1}$ for $i = 1, 2, 3$, and 4 lie in a plane).

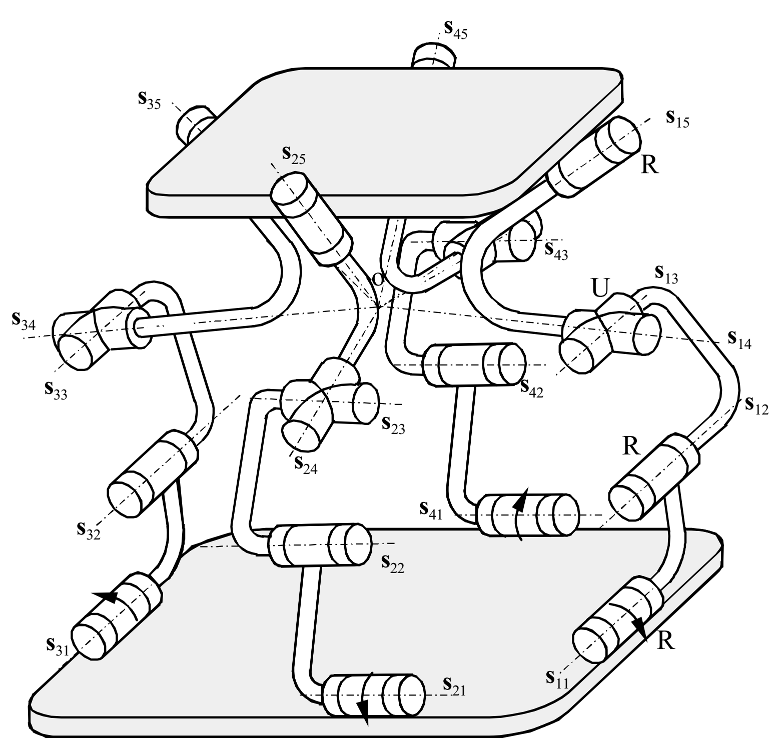

Fig. 3. A 4-DoF manipulator constructed with 4-RRUR F-limbs ($\mathbf{s}_{i1}//\mathbf{s}_{i2}$, $\mathbf{s}_{i1}$ for $i = 1, 2, 3$, and 4 lie in a plane).

Table 2. Feasible C-Limbs

Limb Type	Four-link Limbs	Three-link Limbs	Two-link Limbs
5R	$R_AR_AR_AR_BR_B$	$R_AR_AU_{AB}R_B$	
$4R_1P$	$R_AR_AR_AR_BP$	$R_AR_AU_{AB}P$, $R_AR_AR_AC_B$	
	$R_AR_AR_APR_B$	$R_AR_AC_AR_B$	
	$R_AR_APR_AR_B$	$R_AC_AR_AR_B$, $R_AR_AC_AR_B$, $R_AR_APU_{AB}$	$R_AC_AU_{AB}$
	$R_APR_AR_AR_B$	$C_AR_AR_AR_B$, $R_APR_AU_{AB}$	$C_AR_AU_{AB}$
	$R_BR_AR_AR_AP$	$U_{BA}R_AR_AP$	$U_{BA}R_AC_A$
	$R_AR_AR_BR_BP$	$R_AU_{AB}R_BP$, $R_AR_AR_BC_B$	$R_AU_{AB}C_B$
	$R_AR_AR_BPR_B$	$R_AU_{AB}PR_B$, $R_AR_AC_BR_B$	
	$R_AR_APR_BR_B$	$R_AC_AR_BR_B$	
$3R_2P$	$R_AR_BR_BPP$	$U_{AB}R_BPP$, $R_AR_BC_BP$	$U_{AB}C_BP$
	$R_AR_BPR_BP$	$R_AC_BR_BP$, $U_{AB}PR_BP$, $R_AR_BPC_B$	$U_{AB}PC_B$
	$R_AR_BPPR_B$	$U_{AB}PPR_B$, $R_AC_BPR_B$	
	$R_APR_BR_BP$	$C_AR_BR_BP$, $R_APR_BC_B$, $R_AC_BR_BP$	$C_AR_BC_B$, $R_AC_BC_B$
	$R_APR_BPR_B$	$C_AR_BPR_B$	$C_AC_BR_B$
	$R_APPR_BR_B$	$C_APR_BR_B$, $R_APC_BR_B$	
	$PPR_AR_BR_B$	$PPU_{BA}R_B$, $PC_AR_BR_B$	
	$PR_APR_BR_B$	$PR_AC_BR_B$	
	$PR_AR_BR_BP$	$PU_{AB}R_BP$, $PR_AR_BC_B$	$PU_{AB}C_B$
	$PR_AR_BPR_B$	$PU_{AB}PR_B$	
2R3P	R_AR_BPPP	$U_{AB}PPP$, R_AC_BPP	
	R_APR_BPP	C_AR_BPP	C_AC_BP
	R_APPR_BP	C_APR_BP, R_APC_BP, R_APPC_B	C_APC_B
	R_APPPR_B	C_APPR_B	
	PR_AR_BPP	$PU_{AB}PP$, PR_AC_BP	
	PR_APR_BP	C_APR_BP	

Note that joints arranged in reverse order, known as the kinematic inversion, are not included.

structure such that the five limbs provide only one linearly independent constraint wrench to the moving platform. The limb structure can be taken from Tables 1 or 2, or its kinematic inversion. In what follows, we discuss the assembly conditions of 5-DoF parallel manipulators with F- and C-limbs in turn.

5.1. Parallel Manipulators with Five F-Limbs

For five F-limbs to contribute only one constraint force to the moving platform, the F-limbs must be arranged such that the five constraint forces, one from each F-limb, are co-linear. Since the location of the constraint force provided by an F-limb is determined by the point of intersection of type-1 revolute joints and the direction of the constraint force is determined by the type-2 revolute joints, all five F-limbs must be arranged to satisfy the following conditions:

C11. All type-1 revolute joints in the five F-limbs must intersect at one point.

C12. All type-2 revolute joints in the five F-limbs must be parallel to each other.

A parallel manipulator satisfying the above conditions will have 5-DoF, three rotational DoF and two translational DoF in a plane perpendicular to the constraint force. It follows from conditions C11 and C12 that the only feasible F-limb type is the four-link, (five-jointed) limbs listed in Table 1. Figure 5 shows a 5-$R_2R_2R_2R_1R_1$ parallel manipulator constructed with five F-limbs. The 5-DoF include three rotations about the rotational center O and two translations in a plane perpendicular to the type-2 revolute joints.

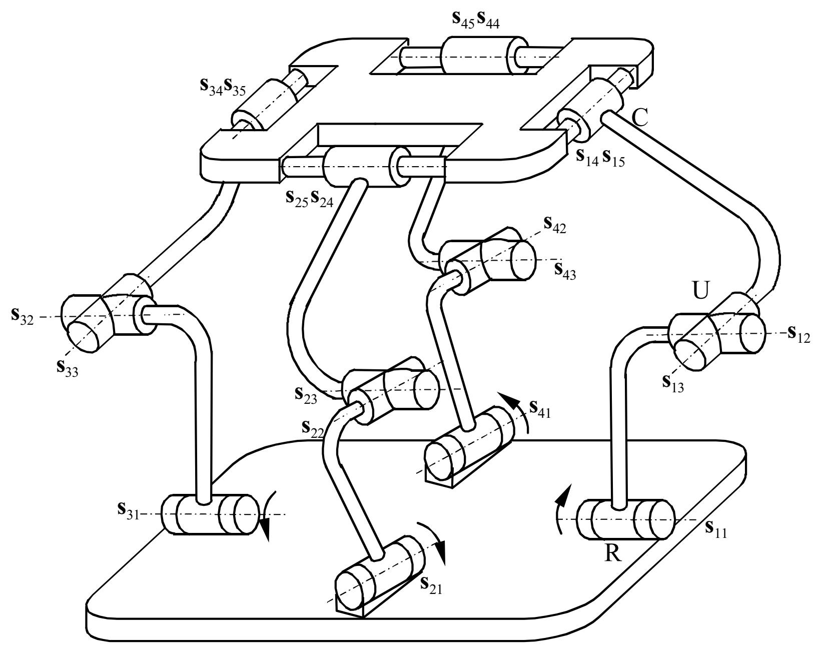

Fig. 4. A 4-DoF manipulator constructed with 4-RUC C-limbs ($\mathbf{s}_{11}//\mathbf{s}_{12}//\mathbf{s}_{31}//\mathbf{s}_{32}$, $\mathbf{s}_{15}//\mathbf{s}_{35}$, $\mathbf{s}_{21}//\mathbf{s}_{22}//\mathbf{s}_{41}//\mathbf{s}_{42}$, $\mathbf{s}_{25}//\mathbf{s}_{45}$).

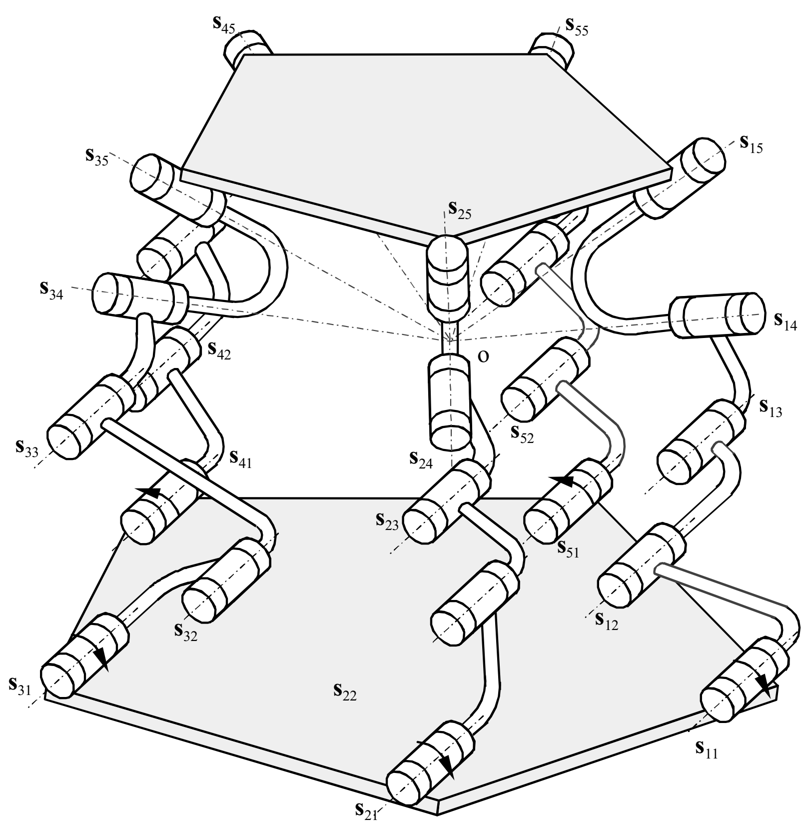

Fig. 5. A 5-DoF manipulator constructed with 5-RRRRR F-limbs ($\mathbf{s}_{11}//\mathbf{s}_{21}//\mathbf{s}_{31}//\mathbf{s}_{41}//\mathbf{s}_{51}$, $\mathbf{s}_{i1}//\mathbf{s}_{i2}//\mathbf{s}_{i3}$ for $i = 1, 2, \ldots, 5$).

5.2. Parallel Manipulators with Five C-Limbs

For a parallel manipulator with five C-limbs to possess 5-DoF, the five constraint couples contributed by the C-limbs must be parallel to each other. Hence, the five C-limbs must satisfy the following conditions:

C13. All the base-connected revolute joints in the five C-limbs must be parallel each other.

C14. All the moving-platform-connected revolute joints of all five C-limbs must also be parallel each other and in a direction different from that of the base-connected joint revolute joints.

The platform wrench system of a parallel manipulator so constructed will contain only one independent constraint couple. Therefore, the parallel manipulator will possess 5-DoF, three translational and two rotational DoF in any direction perpendicular to the constraint couple. Figure 6 shows a 5-RPUR parallel manipulator constructed with five C-limbs. The constraint couple is parallel to the direction defined by $\mathbf{s}_{13} \times \mathbf{s}_{14}$. Hence, rotation of the moving platform about a direction parallel to $\mathbf{s}_{13} \times \mathbf{s}_{14}$ is prohibited.

Due to C13 and C14, the two-link limbs listed in Table 2 are not feasible for constructing 5-DoF parallel manipulators.

6. Singularity Avoidance

We classify the singularities of a low-DoF parallel manipulator into three types. The first type, called *limb singularity*, occurs when the limb twist system degenerates. Under such a condition, the joint screws become linearly dependent and the number of independent wrenches reciprocal to the twist system increases accordingly. As a result, the moving platform loses one or more DoF. The second type, called *platform singularity*, occurs when the overall wrench system of the moving platform degenerates. As a result, the moving platform gains one or more DoF. The third type, called *actuation singularity*, occurs when improper joints are chosen as the driving joints such that when all the actuators are locked, the moving platform still possesses certain DoF. These three types of singularity may occur at certain locations of the workspace or by improper arrangement of the limbs or actuators. Therefore, all three types of singularities should be taken into consideration during the structural synthesis phase.

6.1. Limb Singularity

This type of singularity is similar to the singularity of a serial manipulator. The main cause of limb singularity is due to linear dependence of the joint screws. If a limb singularity occurs and each limb loses 1-DoF, there are two possible changes of the limb wrench system. (1) The additional constraint introduced in each limb is a pure force. The platform wrench system for both 4-DoF and 5-DoF parallel manipulators with either F- and C-limbs will possibly form a six-system resulting in a lock-up of the moving platform. (2) The additional constraint introduced in each limb is a pure couple. Hence, for parallel manipulators constructed with F-limbs, the platform wrench system also possibly becomes a six-system. However, a parallel manipulator constructed with for C-limbs, the platform wrench system can only increase to a three-system and, therefore, the moving platform possesses 3-DoF.

6.2. Platform Singularity

The main reason for platform singularity is due to the linear dependence of the constraint wrenches in the platform wrench system. Since a 5-DoF parallel manipulator consists of only one constraint wrench, the platform singularity is unlikely to occur. There are two possibilities that may introduce platform singularities to a 4-DoF parallel manipulator. (1) When the two constraint forces in the platform wrench system of an F-limbed parallel manipulator become co-linear, the platform wrench system degenerates into a one-system and, consequently, the moving platform gains one translational DoF. (2) When the constraint couples in the platform wrench system of a C-limbed parallel manipulator become parallel each other, the platform wrench system becomes a one-system and the moving platform gains one rotational DoF. Furthermore, platform singularity may occur at certain locations within the workspace of a manipulator. At such locations, the manipulator will lose control.

6.3. Actuation Singularity

In general, any joint in a limb can be selected as an actuated joint. However, due to the existence of some special geometry, an improper choice of actuated joints may result in local mobility. In other words, when all the actuators are locked, some local structure may be mobile. In order to avoid actuation singularities, actuated joints must be chosen such that, when all actuators are locked, the resulting platform wrench system is a six-system. With the actuated joints locked, the dimension of the reciprocal screw system in each limb will increase by 1 (Joshi and Tsai 2002). That is, all screws that are reciprocal to all the joint screws, except for the actuated joint screw, form a two-system. Hence, the number of constraint wrenches in the platform wrench system will increase by four for a 4-DoF parallel manipulator and by five for a 5-DoF parallel manipulator. One way to check for actuation singularity is to calculate the rank of the platform wrench system with all actuators locked. If the rank of such a system is six, the parallel manipulator is free from actuation singularity. Otherwise, the actuated joints should be changed or the structure of a parallel manipulator should be modified.

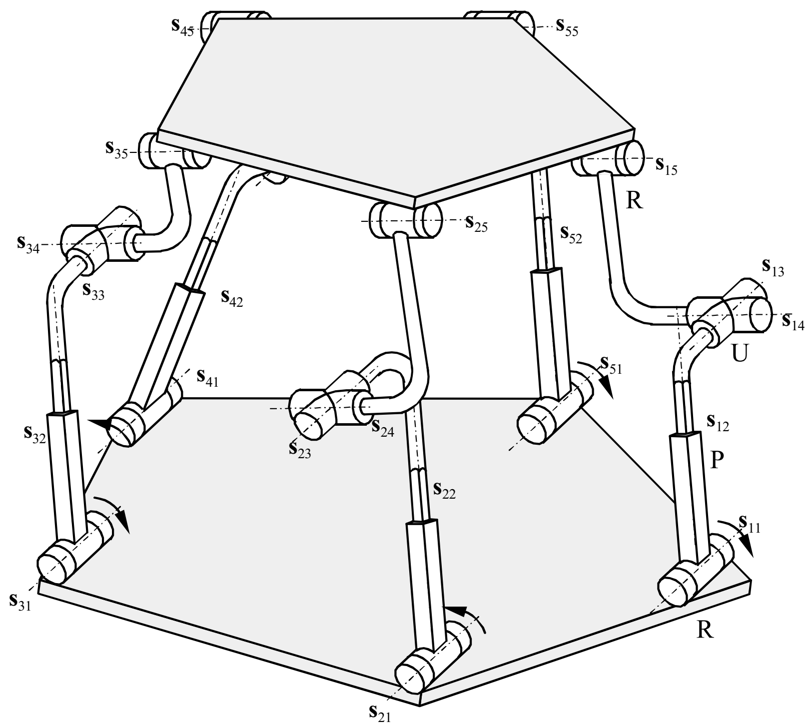

Fig. 6. A 5-DoF manipulator constructed with 5-RPUR C-limbs ($\mathbf{s}_{15}//\mathbf{s}_{25}//\mathbf{s}_{35}//\mathbf{s}_{45}//\mathbf{s}_{55}$, $\mathbf{s}_{11}//\mathbf{s}_{21}//\mathbf{s}_{31}//\mathbf{s}_{41}//\mathbf{s}_{51}$, $\mathbf{s}_{i1}//\mathbf{s}_{i3}$, $\mathbf{s}_{i4}//\mathbf{s}_{i5}$ for $i = 1, 2, \ldots, 5$).

7. Conclusions

In this paper, we present a systematic method for the structural synthesis of a class of 4-DoF and 5-DoF parallel manipulators with identical limb structures. First, the screw theory is used to analyze the constraint conditions for such manipulators to possess a given number of DoF. Then, limb structures that can be used for constructing 4-DoF or 5-DoF parallel manipulators are enumerated according to the reciprocity of the limb twist system and limb wrench system. We discuss the assembly conditions of a parallel manipulator constructed by using identical C- or F-limbs. We show that 4-DoF parallel manipulators constructed by using four F-limbs have one translational and three rotational DoF, and those constructed by using four C-limbs have one rotational and three translational DoF. Five-DoF parallel manipulators constructed by using five F-limbs have three rotational and two translational DoF, and those using five C-limbs have three translational and two rotational DoF.

Acknowledgments

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